Trapdoor Memory-Hard Functions

Benedikt Auerbach

Christoph Günther

Krzysztof Pietrzak

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Memory-hard functions (MHFs)

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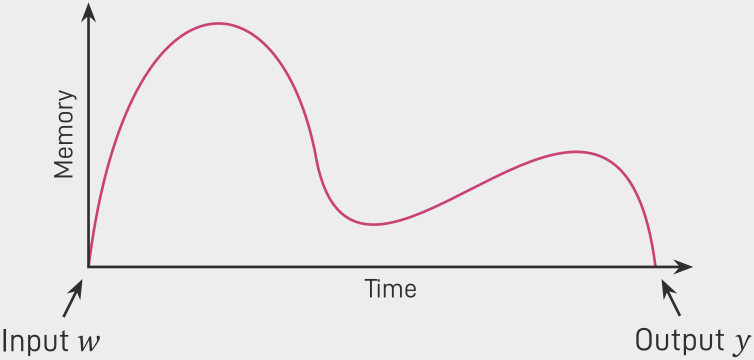
- Scrypt, Argon2 family, DRSample, ...

Memory measure



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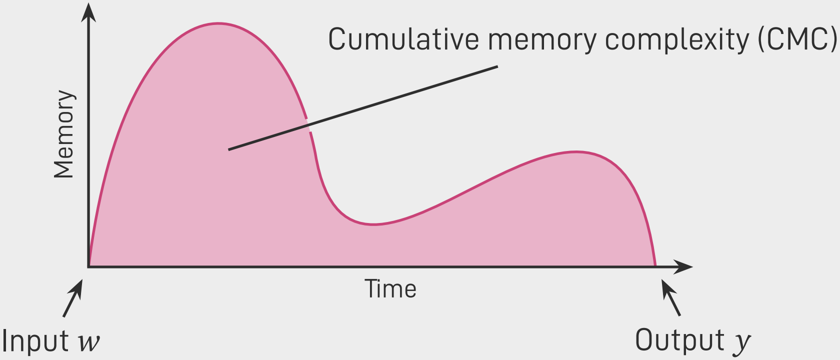

Scrypt (Percival, BSDCan'09)

- Sequential algorithm $\mathrm{Eval}(w) \to y$

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Theorem (Alwen et al., EC'17)

Any (parallel) algorithm evaluating Scrypt has a CMC of $\Omega(n^2 \ell)$ in the random oracle model

Scrypt's Eval

$w = w_0$

Scrypt's Eval

$$w = w_0 \xrightarrow{H(w_0)} w_1$$

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$$w = w_0 \xrightarrow{H(w_0)} w_1 \xrightarrow{H(w_1)} w_2$$

Scrypt's Eval

$$w = w_0 \xrightarrow{H(w_0)} w_1 \xrightarrow{H(w_1)} w_2 \longrightarrow \cdots \longrightarrow w_{n-1}$$

Scrypt's Eval

$$
\begin{tikzcd}
w = w_0 \arrow[r, "H(w_0)"] & w_1 \arrow[r, "H(w_1)"] & w_2 \arrow[r, dashed] & w_{n-1} \arrow[d, "H(w_{n-1})"] \\
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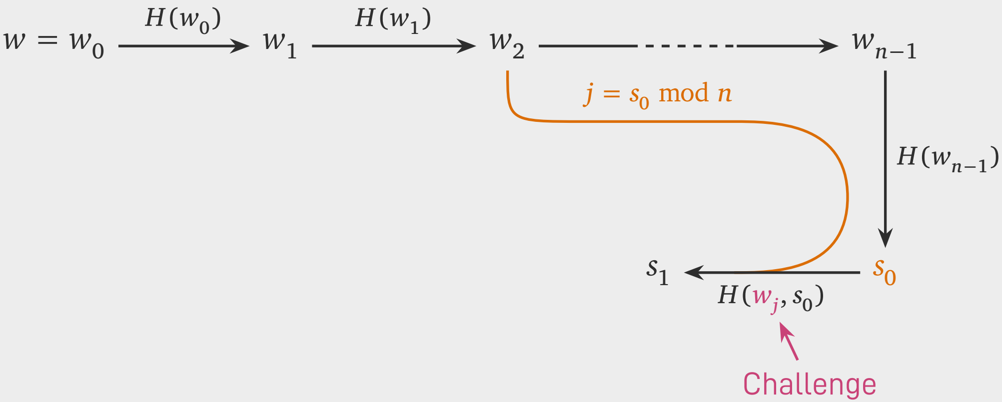
Challenge

Scrypt's Eval



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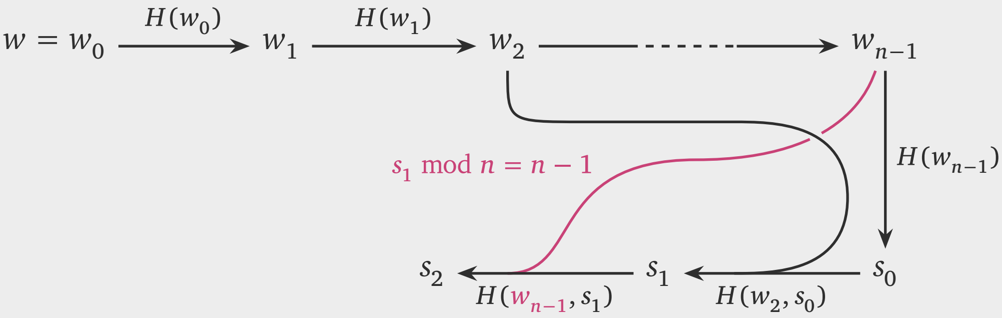

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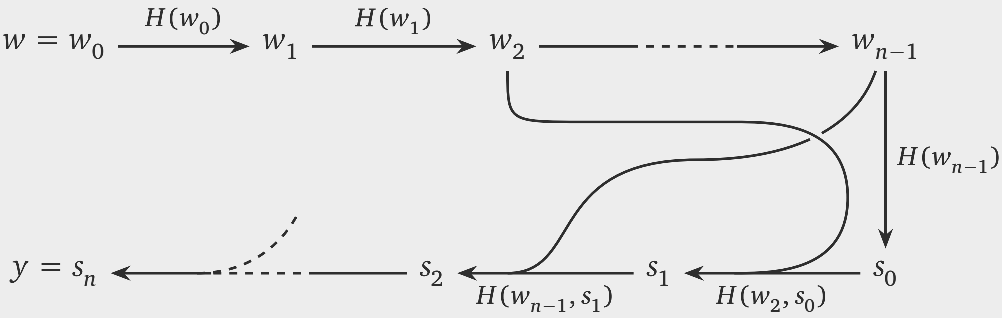

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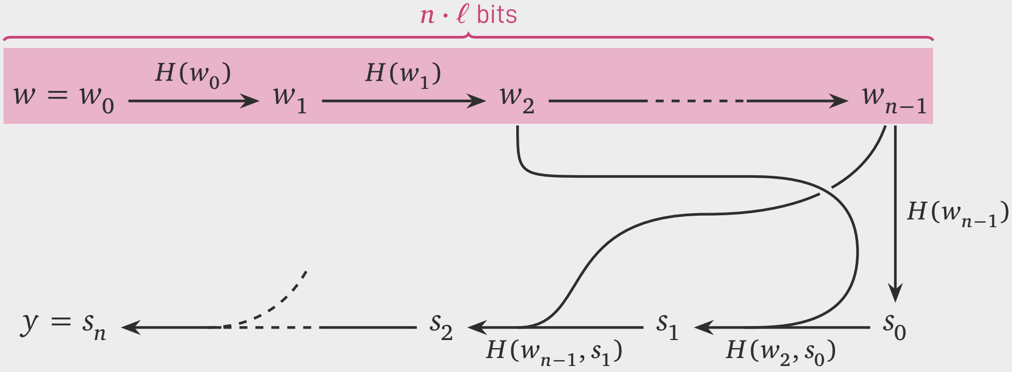

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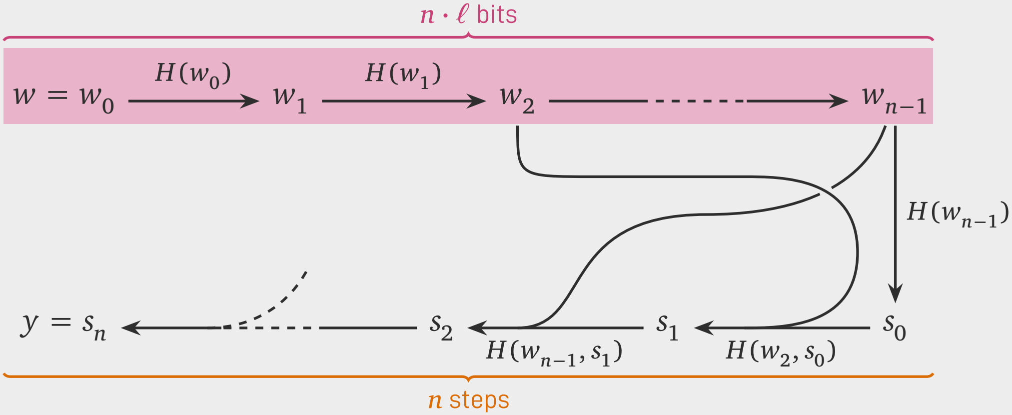

$\text{CMC} \approx n \cdot n \cdot \ell$

Trapdoor memory-hard functions

Algorithms

- $\mathrm{Setup}() \to \mathrm{pp}$
- $\mathrm{Eval}(\mathrm{pp}, w) \to y$

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- $\mathsf{Setup}(\,) \to \mathsf{pp}, \mathsf{td}$
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$\mathsf{Eval}(\mathsf{pp}, w) = \mathsf{TDEval}(\mathsf{pp}, w, \mathsf{td})$

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TD-Efficiency

CMC of TDEval $\ll$ CMC of Eval

Use case: spam prevention

Server

Client

Use case: spam prevention

Server Client

$\mathrm{Eval}(w) = y$

send e-mail + y ←

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$$y \stackrel{?}{=} \mathrm{Eval}(w)$$

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Expensive!

Use case: spam prevention

Server

Client

$\mathrm{Eval}(w) = y$

send e-mail + y

$y \stackrel{?}{=} \mathrm{TDEval}(w, \mathrm{td})$

Cheaper!

Diodon (Biryukov & Perrin, AC'17)

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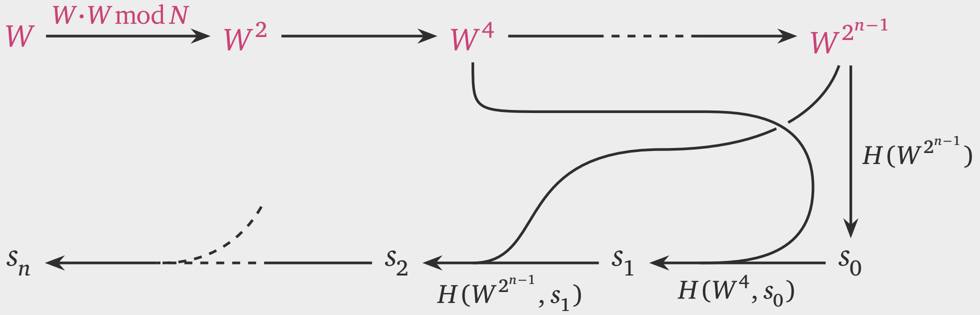
$$\begin{aligned} W_{i+1} &= W_i^2 \bmod N \\ &= W^{2^{i+1}} \bmod N \end{aligned}$$

Diodon's Eval



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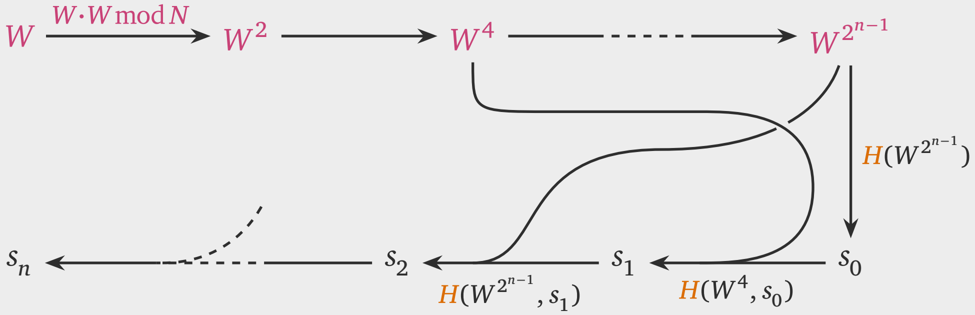

Diodon's TDEval

$$\begin{tikzcd}[column sep=huge, row sep=huge] W \arrow[r, "\text{modular reduction \& exponentiation}"] & W^{2^{n-1} \bmod \varphi(N)} \arrow[d, "H(W^{2^{n-1}})"] \\ & s_0 \end{tikzcd}$$

Diodon’s TDEval

$$\begin{tikzcd}
W \arrow[rr, "\text{mod. red. \& exp.}"] & & W^{2^j} \bmod \varphi(N) \arrow[dl, bend right] & & \\
 & \cdots & s_{i+1} & s_i \arrow[l, "H(W^{2^j}, s_i)"'] & \cdots
\end{tikzcd}$$

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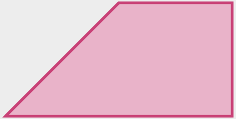

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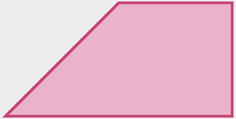

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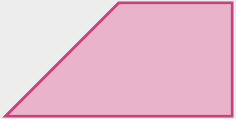

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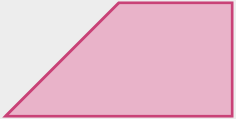

- **Memory-hardness:** **Yes (this work)**

Main result

Theorem

Assuming that factoring is hard, Diodon has a CMC lower bounded by

$$\Omega\left(n^2 \log(N) \cdot \frac{1}{\log n}\right)$$

in the (parallel) random oracle and generic group model

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- Scrypt's proof (Alwen et al., EC'17)
    1. Single-challenge time-memory trade-off
    2. Multi-challenge memory complexity lower bound

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- **Primary hurdle:** Single-challenge trade-off

Single-challenge trade-off

$W \quad W^2 \quad W^4 \quad W^8 \quad W^{16} \quad W^{32}$

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Challenge $j = 2$

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$\mathscr{A}$

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\mathscr{A} \\
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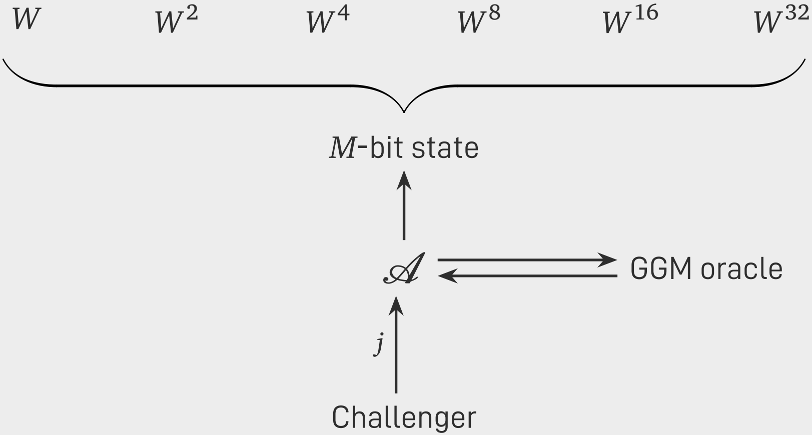

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$\mathscr{A}$ GGM oracle

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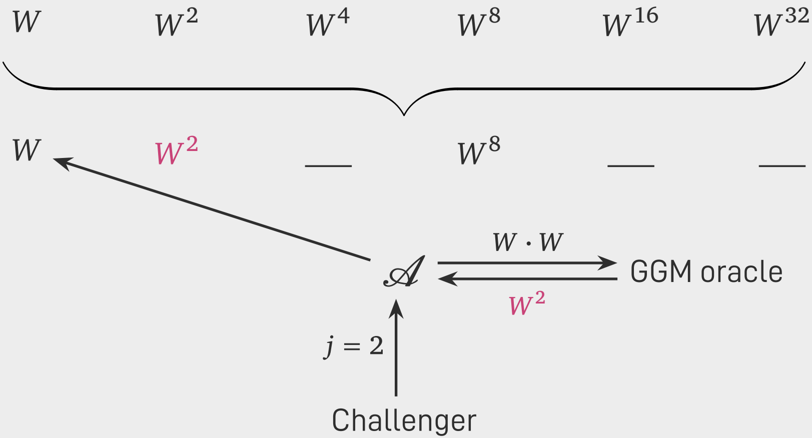

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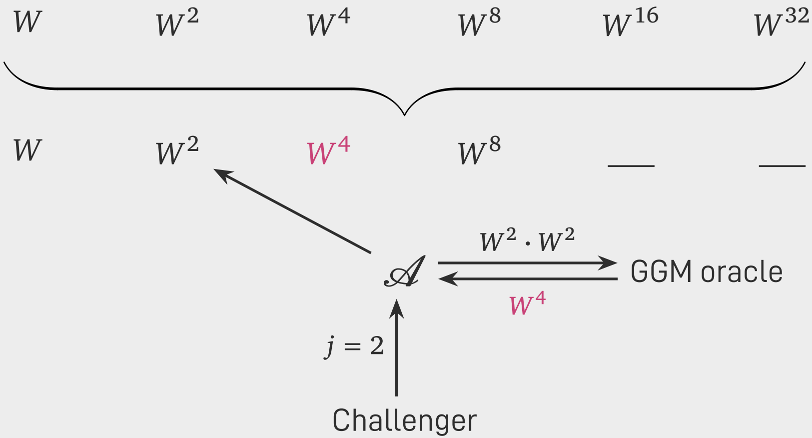

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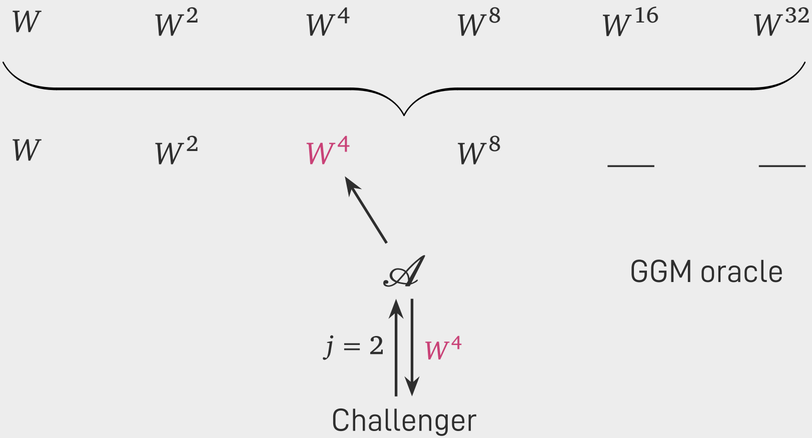

Single-challenge intuition

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- ...but $1/3$ of the challenges require 2 queries!
- Intuitively: $M / \log N$ equidistant group elements offers good trade-off
- **We prove that one cannot do much better**

Single-challenge bound

- M-bit state
- Challenge $j \in \{0, \ldots, n-1\}$ requires t_j GGM queries

Time-memory trade-off

$$\Pr_j\left[t_j \gtrsim \frac{n}{2 \cdot M / \log N} \cdot \frac{1}{\log n}\right] \geq \frac{1}{2}$$

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Equidistant strategy

Not tight?

How to prove the bound?

Contradiction: $\mathscr{A}$ answers

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2. Most t_j small and $\varphi(N)$ hidden $\implies$ $\vec{x}$ has many entries

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   $\implies$ $\vec{x}$ has many entries
3. **Case 1:** $\vec{x}$ has few entries

   $\implies$ $\mathscr{A}$ knows $\varphi(N)$

   $\implies$ Factor N $\lightning$
4. **Case 2:** $\vec{x}$ has many entries

   $\implies$ $\vec{x}$ contains a lot of info about the GGM oracle

   $\implies$ Compress to M bits $\lightning$

Conclusion

Contribution

Diodon's CMC lower bounded by

$$\Omega\left(n^2 \log(N) \cdot \frac{1}{\log n}\right)$$

proving it memory-hard

Open questions

- Tight bound (no $1/\log n$)
- TMHF saving on time and memory
- TMHF for other MHF flavors

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https://eprint.iacr.org/2024/312